Harmonic oscillator partition function from the path integral

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We take the Euclidean action to be $\dot{X}^2/2 + E^2X^2/2$. Directly using the energy spectrum, we can compute the partition function:

$$\begin{aligned} Z_{\text{ho}}(\beta) &= e^{-\beta E/2} + e^{-3\beta E/2} + \dots \\ &= \frac{1}{2\sinh \beta E/2}\,. \end{aligned}$$

To connect to the path integral language, we write this in a more suggestive way using the infinite product formula for sinh.

$$\begin{aligned} Z_{\text{ho}}(\beta) = \frac{1}{2\sinh \beta E/2} &= \frac{1}{\beta E}\prod_{n=1}^{\infty}\left(1+\frac{\beta^2E^2}{4\pi^2n^2}\right)^{-1} \\ &= \frac{1}{\beta E}\prod_{n=1}^{\infty}\frac{\omega_n^2}{\omega_n^2+E^2}\,. \end{aligned} \tag{1}$$

In the the path integral language, the terms with $n \neq 0$ in the infinite product come from the nonzero Matsubara modes (after dividing by the answer for the free particle with $E = 0$). The only thing left to explain is why the $n = 0$ mode gives $1/(\beta E)$. The path integral weight for the $n = 0$ mode is

$$\exp -\beta \times \frac{1}{2}E^2X_0^2\,.$$

Integrating this over X_0 will give $\sqrt{2\pi/\beta E^2}$. Thus the $1/E$ comes out correct, but the factor of 2π and β needs to be fixed.

The trouble with the $n = 0$ mode is that we cannot figure out the proper measure factor multiplying dX_0 by comparing it to the free answer because the zero mode when $E = 0$ is not Gaussian suppressed, and gives ∞. One nice way to fix this is to consider the real-time propagator for the free particle which is given by

$$\langle x_f | e^{-iH_{\rm free}t} | x_i \rangle = \sqrt{\frac{1}{2\pi i t}}\,\exp - \frac{(x_f - x_i)^2}{2it}\,.$$

The prefactor here is fixed by the requirement that as $t \to 0$, the propagator must approach $\delta(x_f - x_i)$. To go to Euclidean signature, we have to replace it with β. In order to get the partition function, we have to take the trace. This involves setting $x_f = x_i$, which gets rid of the exponential factor. The integral over x gives the divergence proportional to the "volume of the target space".

$$Z_{\rm free}(\beta) = \frac{1}{\sqrt{2\pi\beta}} \times ({\rm vol}\ X)\,.$$

This means that

$$\begin{aligned}\left(\int dX_0\right) \prod_{n>0} \frac{1}{\omega_n^2} &= \frac{1}{\sqrt{2\pi\beta}} \times ({\rm vol}\ X), \quad \text{or} \\ 1 &= \frac{1}{\sqrt{2\pi\beta}} \prod_{n>0} \omega_n^2\end{aligned}$$

We take the naive answer for the harmonic oscillator and 'insert one':

$$\begin{aligned}Z_{\rm ho}(\beta) &= \sqrt{\frac{2\pi}{\beta E^2}} \prod_{n>0} \frac{1}{\omega_n^2 + E^2} \\ &= \sqrt{\frac{2\pi}{\beta E^2}} \frac{1}{\sqrt{2\pi\beta}} \prod_{n>0} \frac{\omega_n^2}{\omega_n^2 + E^2}.\end{aligned}$$

The last expression is precisely what was needed, see (1).